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General Relativity, the Massless Scalar Field, and the Cosmological Constant

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For gravity coupled to a neutral, massless scalar field, Wyman suggested a method of solution in power series valid provided the scalar field depends only on time. In this work we generalize his approach to nonzero cosmological constant.

1. INTRODUCTION

General relativity couples gravity with all fields. In particular, general covariance determines the relevance of the metric of space-time in the equations of the fields, and these, in view of their energy-momentum content, constitute the external sources of gravity.

Bergmann and Leipnik (1957) sought solutions for the coupling of gravity with a neutral, massless scalar field. They assumed a static line element with spherical symmetry. Several authors have since dealt with various aspects of the problem; see Frøyland (1982) for some important references.

Under the assumptions of Bergmann and Leipnik, Wyman (1981) suggested a new coordinate system which allows the integration of the field equations in an almost trivial manner, provided the scalar field is time independent. He further suggested a method for searching for a solution in a power series, provided the scalar field is only time dependent. However, all these results are based on a vanishing cosmological constant Λ.

The purpose of this work is to study the coupling of a scalar neutral massless field with gravity allowing a nonnull value for Λ in Einstein's equations. In particular, we shall consider a class of solutions corresponding to a scalar field that only depends on time.

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The plan of this work is as follows: In Section 2 we formulate the field equations for the coupled system. In Section 3 we search for Schwarzschild-type solutions for the coupled equations. For this purpose we find a natural extension of the power series method for $\Lambda \neq 0$. A brief summary of the results is presented in Section 4.

2. THE FIELD EQUATIONS

The most general case of Einstein's equations is

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu} \tag{1}$$

where $R_{\mu\nu}$ denotes the Ricci tensor; R denotes the curvature scalar, constructed from the metric tensor $g_{\mu\nu}$ and its derivatives. Λ must be a constant, called the cosmological constant. κ denotes the gravitational coupling constant, and $T_{\mu\nu}$ denotes the energy-momentum tensor associated with external gravitation sources.

Let ϕ (the neutral massless scalar field) be the only external source of the gravitational field. The associated $T_{\mu\nu}$ minimally coupled to gravity is defined as

$$\kappa T_{\mu\nu} = \Omega(\phi_{,\mu}\phi_{,\nu} - \tfrac{1}{2} g_{\mu\nu} g^{\alpha\beta} \phi_{,\alpha}\phi_{,\beta}) \tag{2}$$

Here a comma denotes the partial derivative and Ω is a positive constant. The equation of motion for ϕ (minimally coupled) is given by

$$\phi_{;\alpha}^{;\alpha} = 0 \tag{3}$$

where semicolon denotes a covariant derivative.

The field equations (1) allow R to be expressed in terms of Λ and T^{α}_{α}, the trace of $T_{\mu\nu}$, which may be calculated from (2). Substituting these results in (1), we reduce Einstein's equations to the equivalent form

$$R_{\mu\nu} = -\Omega \phi_{,\mu}\phi_{,\nu} + \Lambda g_{\mu\nu} \tag{4}$$

Finally, we observe that all physically acceptable solutions to (3) correspond to an energy density T_{00} positive in a local Lorentzian frame, defined by

$$\begin{aligned} &g_{00} = -g_{11} = -g_{22} = -g_{33} = 1 \\ &g_{\mu\nu} = 0, \qquad \mu \neq \nu \end{aligned} \tag{5}$$

3. SOLUTION OF THE EQUATIONS

In the static case with spherical symmetry, the Schwarzschild-type line element is given by

$$ds^2 = e^{\nu}\, dt^2 - e^{\lambda}\, dr^2 - r^2\, d\Sigma^2 \tag{6}$$

where

$$d\Sigma^2 = d\theta^2 + \sin^2\theta\, d\varphi^2 \tag{7}$$

both ν and λ being only functions of r. Similarly, following Wyman (1981), we shall suppose that the field ϕ depends at most on t and r. Let $\dot{\phi}$ and ϕ' denote the partial derivatives of ϕ with respect to t and r, respectively. Then $\phi_{,\mu}$ takes the form

$$\phi_{,\mu} = (\dot{\phi}, \phi', 0, 0) \tag{8}$$

Due to the spherical symmetry assumed in (6), the field equations (4) imply that $\dot{\phi}$ and ϕ' are independent of the t variable. Clearly, $\phi = \phi(t, r)$ may depend explicitely on t.

Under our symmetry assumptions, the only nonnull components of $R_{\mu\nu}$ are (Adler, et al., 1975, p. 464)

$$\begin{aligned}
R_{00} &= -e^{\nu-\lambda}\left(\frac{\nu''}{2} - \frac{\nu'\lambda'}{4} + \frac{\nu'^2}{4} + \frac{\nu'}{r}\right)\\
R_{11} &= \frac{\nu''}{2} - \frac{\nu'\lambda'}{4} + \frac{\nu'^2}{4} - \frac{\lambda'}{r}\\
R_{22} &= e^{-\lambda}\left(1 + \frac{\nu' r}{2} - \frac{\lambda' r}{2}\right) - 1\\
R_{33} &= R_{22}\sin^2\theta
\end{aligned} \tag{9}$$

Then, given the structure of $R_{\mu\nu}$, and considering the definitions of $\phi_{,\mu}$ and $g_{\mu\nu}$ given in (8) and (6), respectively, we obtain the field equations

$$-e^{\nu-\lambda}\left(\frac{\nu''}{2} - \frac{\nu'\lambda'}{4} + \frac{\nu'^2}{4} + \frac{\nu'}{r}\right) = -\Omega\dot{\phi}^2 + \Lambda\, e^{\nu} \tag{10}$$

$$\frac{\nu''}{2} - \frac{\nu'\lambda'}{4} + \frac{\nu'^2}{4} - \frac{\lambda'}{r} = -\Omega\phi'^2 - \Lambda\, e^{\lambda} \tag{11}$$

$$e^{-\lambda}\left(1 + \frac{\nu' r}{2} - \frac{\lambda' r}{2}\right) - 1 = -\Lambda r^2 \tag{12}$$

$$\dot{\phi}\phi' = 0 \tag{13}$$

On the other hand, (3) takes the form

$$\frac{\partial}{\partial r}(r^2\, e^{(\nu-\lambda)/2}\phi') - \frac{\partial}{\partial t}(r^2\, e^{(\lambda-\nu)/2}\,\dot{\phi}) = 0 \tag{14}$$

Equation (13) imples that $\phi' \neq 0$ or $\dot{\phi} \neq 0$. These cases must be dealt with separately. In this work we shall only consider the simplest case, $\dot{\phi} \neq 0$, which allows an immediate integration of (14).

If $\dot{\phi} \neq 0$, we see from equation (13) that $\phi' = 0$. Therefore, in view of equation (14), we have

$$\dot{\phi} = \text{const} \tag{15}$$

In what follows, $\dot{\phi}$ shall denote a constant of integration.

Equations (10)-(12) imply

$$\nu' + \lambda' = \Omega \dot{\phi}^2 r\, e^{\lambda - \nu} \tag{16}$$

$$\nu' - \lambda' = 2[(1 - \Lambda r^2)\, e^{\lambda} - 1]/r \tag{17}$$

In the particular case $\Lambda = 0$, Wyman (1981) suggested a method of solution in power series for equations (16) and (17). We now proceed to show the extension of his method for the general case $\Lambda \neq 0$. The basic fact is that in the absence of explicit solutions for $g_{\mu\nu}$ in this system of coordinates, the system (16), (17) allows the calculation of the Taylor expansions for the solutions at any point of space. In particular, we shall develop the procedure at the point $r = 0$. We shall simplify the calculations by defining the new variable

$$x = \tfrac{1}{2} \dot{\phi}^2 \Omega r^2 \tag{18}$$

This change of variable reduces (16) and (17) to

$$\nu' + \lambda' = e^{\lambda - \nu} \tag{19}$$

$$\nu' - \lambda' = [(1 - \varepsilon x)\, e^{\lambda} - 1]/x \tag{20}$$

where the prime denotes differentiation with respect to x; the ε parameter has been defined as

$$\varepsilon = 2\Lambda/\Omega \dot{\phi}^2 \tag{21}$$

Let $\nu(x)$ and $\lambda(x)$ be regular at $x = 0$. Then $\lambda(0) = 0$. In particular, we shall impose $\nu(0) = 0$. Under these conditions, equations (19) and (20) become

$$\nu'(0) + \lambda'(0) = 1 \tag{22}$$

$$\nu'(0) - \lambda'(0) = \lim_{x \to 0} \frac{(1 - \varepsilon x)\, e^{\lambda} - 1}{x} = \lambda'(0) - \varepsilon \tag{23}$$

Solving (22) and (23) for $\nu'(0)$ and $\lambda'(0)$, we obtain

$$\nu'(0) = (2 - \varepsilon)/3 \tag{24}$$

$$\lambda'(0) = (1 + \varepsilon)/3 \tag{25}$$

Differentiating equations (19) and (20) with respect to x, we find, in the limit $x \to 0$, the equations for $\nu''(0)$ and $\lambda''(0)$. Using (24) and (25), we obtain

$$\nu''(0)+\lambda''(0)=\frac{2\varepsilon-1}{3} \tag{26}$$

$$\nu''(0)-\tfrac{3}{2}\lambda''(0)=-\frac{\varepsilon(1+\varepsilon)}{3}+\frac{(1+\varepsilon)^2}{18} \tag{27}$$

Therefore, $\nu''(0)$ and $\lambda''(0)$ are given by

$$\nu''(0)=\frac{-8+14\varepsilon-5\varepsilon^2}{45} \tag{28}$$

$$\lambda''(0)=\frac{-7+16\varepsilon+5\varepsilon^2}{45} \tag{29}$$

Now we are able to express the first terms of the Taylor expansions for $\nu(x)$, $\lambda(x)$, e^{ν}, and e^{λ} at $x=0$. The results are

$$\nu(x)=\frac{2-\varepsilon}{3}x-\frac{8-14\varepsilon+5\varepsilon^2}{90}x^2+\cdots \tag{30}$$

$$=\frac{2-\varepsilon}{6}\dot{\phi}^2\Omega r^2-\frac{8-14\varepsilon+5\varepsilon^2}{360}\dot{\phi}^4\Omega^2 r^4+\cdots \tag{31}$$

$$\lambda(x)=\frac{1+\varepsilon}{3}x-\frac{7-16\varepsilon-5\varepsilon^2}{90}x^2+\cdots \tag{32}$$

$$=\frac{1+\varepsilon}{6}\dot{\phi}^2\Omega r^2-\frac{7-16\varepsilon-5\varepsilon^2}{360}\dot{\phi}^4\Omega^2 r^4+\cdots \tag{33}$$

$$e^{\nu}=1+\frac{2-\varepsilon}{3}x+\frac{2-\varepsilon}{15}x^2+\cdots \tag{34}$$

$$=1+\frac{2-\varepsilon}{6}\dot{\phi}^2\Omega r^2+\frac{2-\varepsilon}{60}\dot{\phi}^4\Omega^2 r^4+\cdots \tag{35}$$

$$e^{\lambda}=1+\frac{1+\varepsilon}{3}x-\frac{2-26\varepsilon-10\varepsilon^2}{90}x^2+\cdots \tag{36}$$

$$=1+\frac{1+\varepsilon}{6}\dot{\phi}^2\Omega r^2-\frac{1-13\varepsilon-5\varepsilon^2}{180}\dot{\phi}^4\Omega^2 r^4+\cdots \tag{37}$$

A new aspect of our results is that the Taylor coefficients calculated for the expansions (30), (32), (34), and (36) are polynomial functions of the ε parameter, with all its roots real and nonull.

In particular, equations (2) and (15) imply that the T_{00} component of the energy-momentum tensor is constant and given by

$$\kappa T_{00} = \Omega \dot{\phi}^2/2 \tag{38}$$

Then, ε, defined by (21), may be expressed as

$$\varepsilon = \Lambda/\kappa T_{00} \tag{39}$$

Then ε and the Taylor coefficients in the expansions (30), (32), (34), and (36) remain unaltered when Λ and T_{00} are rescaled by the same factor.

Finally, we remark that our expressions for ν, λ, e^{ν}, and e^{λ} include, as special cases, the expressions obtained by Wyman (1981) for $\varepsilon = 0$ ($\Lambda = 0$).

4. FINAL COMMENTS

We have extended the method of solution as power series to the most general case of Einstein's equations, including the cosmological term, when a scalar, neutral, massless field is the only external source of gravity. The results we have obtained include, as a particular case, those reported previously for $\Lambda = 0$.

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